DECAY OF OSCILLATIONS OF A SPHERE AND CYLINDER IN A LIQUID—A NEW METHOD OF DETERMINING VISCOSITY

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ABSTRACT. With the help of the dissipation function, it is deduced that the modulus of decay of torsional oscillations of a sphere of radius 'r' and moment of inertia I about a diameter, in a liquid of viscosity η is given by $8\pi r^3/I$ and that of a cylinder of radius a immersed in a liquid, to a depth l, contained in a concentric cylinder of radius b is given by $2\pi\eta\, l \dfrac{a^2b^2}{I(b^2-a^2)}$ where I is the moment of inertia of the cylinder about its axis of suspension. By knowing the logarithmic decrement the viscosity of a liquid can be found. The experimental values are found to agree with standard values.

INTRODUCTION

Among the methods of finding the viscosity of liquids are Poiseuille's capillary tube method, rotating cylinder (Couettee 1890, Searle 1912) and Stoke's (1851) method for highly viscous liquids. Measurement of modulus of decay of liquids in vessels or of solids in liquids was used by Helmholtz (1850) and Meyer (1891) for the determination of viscosity. But these require corrections which are complicated. A comparative method of measuring viscosity was first given by Mennerette (1911), who studied the decay of oscillations of liquids in a U-tube. By making use of the dissipation function Venkataraman (1933) deduced an expression for the modulus of decay in the above case and thus made it an absolute method. Christopherson and others (1938) made a study of the forced oscillations of a liquid in a U-tube and after elaborate mathematical analysis obtained expressions for the viscosity. Recently Choudhary and Trivedi (1961) described a method by which the viscosities can be compared by noting the damping of rotation of a symmetrical body in a liquid. This method is, however, useful only in the case of a very viscous liquids. In the concentric cylinder method, it is necessary to maintain a uniform motion of the external cylinder to get a constant deflection of the inner cylinder. This difficulty can be avoided by measuring the modulus of decay of the oscillations of the inner cylinder when it is twisted through a small angle and allowed to execute free oscillations. The same method can be used in the case of the sphere. This is what is attempted in this investigation.

THEORY

The modulus of decay of oscillations can be found from the dissipation function which was first given by Stokes (1851) and used by Lamb (1911) in the solution of some problems in hydrodynamics.

The dissipation function suitable for the rotational oscillations of a sphere in a liquid can be written as

$$2F = \eta\iiint(\omega^2_x+\omega^2_y+\omega^2_z)dx\,dy\,dz \qquad \dots \quad (1)$$

where η is the viscosity of the liquid, ω_x, ω_y, ω_z are the components of the angular velocity. The integration is to be taken over the whole volume of the liquid. The dissipation is equal to the rate of decay of the total energy ϵ i.e., $d\epsilon/dt$. In the case of a sphere, suspended in a liquid by a long thin wire attached to a chuck fixed to a stand, the kinetic energy $=\frac{1}{2}I\omega^2$ where ω is the angular velocity of the sphere and I is the moment of inertia of the sphere about an axis passing through its centre. If the angular velocity is variable, it is equal to $\frac{1}{2}I(d\theta/dt)^2$ where θ is the angular displacement. Since the sphere is executing harmonic oscillations, its total energy is twice this, i.e.,

$$I\left(\frac{d\theta}{dt}\right)^2 \qquad \dots \quad (2)$$

Sphere in an infinite liquid

The dissipation function for a sphere in an infinite liquid can be easily found. In the case of a sphere rotating uniformly about its diameter in a liquid of viscosity η the moment of the couple acting on it is given by (Lamb 2, 1916)

$$C' = 8\pi\eta a^3\omega \qquad \dots \quad (3)$$

The dissipation function given in (1) is therefore

$$2F = 8\eta a^2\omega^2 \qquad \dots \quad (4)$$

when the angular velocity is varying, it is given by $8\pi\eta a^3(d\theta/dt)^2$. Equating the rate of decay of total energy to dissipation, we get

$$\frac{d}{dt}\left[I\left(\frac{d\theta}{dt}\right)^2\right] = -8\pi\eta a^3\left(\frac{d\theta}{dt}\right)^2 \qquad \dots \quad (5)$$

writing x for $\left(\frac{d\theta}{dt}\right)^2$, we have

$$I\left(\frac{dx}{dt}\right) = -8\pi\eta a^3 x$$

Integrating this we get

$$\log x = -\frac{8\pi\eta a^3 t}{I} + \text{const} \qquad \dots \quad (6)$$

If we put the boundary condition $x = x_0$ when $t = 0$, the above becomes

$$x = x_0 \exp[-8\pi a^3 \eta t/I]. \quad \dots (7)$$

Substituting $(d\theta/dt)^2$ for x, equation (7) can be written as

$$\frac{d\theta}{dt} = \theta' \exp[-4\pi\eta a^3 t/I];$$

when this is integrated we obtain

$$\theta = \theta_0 \exp[-4\pi\eta a^3 t/I]$$

where θ_0 is a new constant, thus the logarithmic decrement is

$$-4\pi\eta a^3/I \quad \dots (8)$$

Concentric cylinders

In the case of a cylinder of radius a executing rotational oscillations in a liquid contained in another concentric cylinder of radius b the viscous couple on the inner cylinder, when the outer cylinder rotates with uniform angular velocity ω is given by (Lamb 3, 1916).

$$4\pi\eta \frac{a^2b^2}{b^2-a^2} l\omega \quad \dots (9)$$

Where l is the length of the immersed portion of the cylinder in the liquid. As before, the dissipation function in this case is

$$4\pi\eta \frac{a^2b^2}{b^2-a^2} l\omega^2 \quad \dots (10)$$

Thus equating the rate of decay of total energy to dissipation we get an equation corresponding to (5)

$$\frac{d}{dt}\left[I\left(\frac{d\theta}{dt}\right)^2 = -4\pi\eta l \frac{a^2b^2}{b^2-a^2}\left(\frac{d\theta}{dt}\right)^2.\right.$$

Integrating the equation as in the previous case we get

$$\theta = \theta_0 \exp\left[-\frac{2\pi\eta l}{I} \cdot \frac{a^2b^2}{b^2-a^2} t\right].$$

Thus the modulus of decay is

$$-\frac{2\pi\eta l a^2b^2}{I(b^2-a^2)} \quad \dots (11)$$

Equation (8) and (11) form the basis of the measurement of viscosity by finding the modulus of decay of a sphere in a liquid and a cylinder in a liquid contained in a concentric cylinder.

EXPERIMENTAL ARRANGEMENT

For finding the logarithmic decrement a steel sphere is suspended by a long thin wire whose other end is fixed to a heavy stand. A small mirror is fixed to the wire to note the deflections by means of a telescope and scale arrangement. The length is measured by a cathetometer. The entire set-up is shielded from draught by enclosing it in a glass case. The sphere is fully immersed in a liquid contained in a beaker. A thermoregulator was used to keep the temperature constant.

The sphere is given an angular displacement and the oscillations are observed with the help of a telescope and scale. The maximum time taken for a set of reading is 5 or 6 minutes. From the first and final quarter swings (to the same side) the logarithmic decrement is computed. If α_1 and α_{n+1} are the first and $(n+1)$ the deflections, say, to the right side, then the logarithmic decrement is given by

$$\lambda = \frac{1}{n} \log \frac{\alpha_1}{\alpha_{n+1}}$$

or, if logarithm to the base 10 are used

$$\lambda = \frac{2.3026}{n} \log_{10} \frac{\alpha_1}{\alpha_{n+1}} \qquad \ldots \quad (12)$$

from relation (8) we get

$$\lambda = \tfrac{1}{2} K \eta I \qquad \ldots \quad (13)$$

where $K - 4\pi\gamma^3/I$. Substituting $\frac{2}{5}Mr^2$ for I, where M is the mass of thes phere, we get

$$K = \frac{10\pi r}{M}$$

The coefficient of viscosity is therefore, given from (12) and (13) by

$$\eta = \frac{4.6052}{\eta KT} \log_{10} \frac{\alpha_1}{\alpha_{n+1}} \qquad \ldots \quad (14)$$

where T is the periodic time. The time for 25 oscillations was measured to 0.2″. The time is of the order of 5 or 6 minutes and the time for one oscillation is calculated and is given to two decimal places.

The results are given in Table I.

Cylinder in a liquid contained in a coaxial cylinder

Just as in the above case the inner cylinder, of radius 'a' is suspended by a wire and is immersed in a liquid contained in an outer cylinder of radius '*b*',

TABLE I

Logarithmic decrement and viscosity of liquids with a sphere (using a sphere 1″ in diameter).

Sl. No.	Liquid	Period T (in seconds)	Logarithmic Decrement	Coefficient of viscosity in centipoises	
				Expt. at 25°C.	Standard at 25°C.
1.	Toluene	11.01	1.519	0.525	0.525
2.	Benzene	11.00	1.794	0.622	0.604
3.	Water	11.28	2.624	0.886	0.893
4.	Carbon tetra chloride	11.28	2.738	0.922	0.905
5.	Acetic Acid	11.30	3.442	1.160	1.113
6.	Olive Oil*	1.80	21.640	66.0	68.0
7.	Mobiloil SAE 40*	1.84	67.370	217.0	214.0

*using a sphere 2.0″ in diameter.

We get for the angular displacement as shown above

$$\theta = \theta_0 \exp\left[-\frac{2\pi a^2 b^2 \eta l}{(b^2-a^2)I}\cdot t\right]$$

The decrement for one oscillation is therefore

$$\lambda = \frac{2\pi a^2 b^2 \eta l}{(b^2-a^2)I}\cdot\frac{T}{2} \qquad \text{... (13a)}$$

where T is the periodic time.

The coefficient of viscosity is given from (13a) and (12) by the relation

$$\eta = \frac{2.3026}{\eta T K'} \log_{10}\frac{\alpha_1}{\alpha_{n+1}} \qquad \text{... (15)}$$

where $$K' = \frac{\pi a^2 b^2}{b^2-a^2}\cdot\frac{l}{I}$$

The experimental set-up is the same as in the previous case; the sphere is replaced by the cylinder. The outer vessel, containing the liquid is also a cylinder with a perfectly flat bottom. The cylinder used here is a brass cylinder. In order to eliminate the effects of viscous forces on the ends of the suspending cylinder another cylinder of the same diameter as the suspending cylinder is placed right below and the length of the wire is so adjusted that the space between the two is less than 0.5mm. and the top end of the suspending cylinder is about 5mm. above the surface of the liquid. The outer cylinder is a glass cylinder so as to enable one to measure the length of the cylinder immersed in the liquid and also to check the parallelism of the suspending cylinder and the guard cylinder.

The temperature of the liquid was kept constant at 30°C (which happens to be the room temperature during the experiment), with the help of the thermo-regulator. The procedure is the same as in the case of a sphere. The lengths are measured by a reading microscope reading up to 0.01 mm.

Results : The results, after using relation (5), are given in Table II.

TABLE II

Logarithmic decrement and viscosity of liquids with concentric cylinders (using a cylinder of diameter of 2.55 cms).

Sl. No.	Liquid	Length in cms.	Period 'T' in seconds	Logarithmic decrement	Coeff. of viscosity in centipoises	
					Expt. at 30°c.	Std. value at 30°c.
1.	Acetic Acid	2.20	25.30	3.608	1.039	1.040
2.	Carbon tetra chloride	2.25	25.40	2.945	0.826	0.848
3.	Benzene	2.30	25.40	2.052	0.563	0.561
4.	Ethyl Acetate	2.20	25.40	1.417	0.406	0.407
5.	Olive Oil*	2.62	9.00	35.0	49.60	52.00
6.	Mobil oil SAE 40*	2.48	9.20	71.4	208.0	212.1
7.	,, ,, 30*	2.52	9.15	95.9	278.0	290.0

*using cylinder of diameter of 3.82 cms.

TABLE III

Variation of log. decrement with depth of liquid between cylinders

Liquid	Spacing in mm	Length of cylinder immersed	log. decrement $\times 10^{-2}$	Viscosity in CP at 30°c.	Standard value at 30°c.
	5.45	2.22	3.642	1.049	
Acetic Acid	4.18	2.20	3.635	1.047	1.040
	2.40	2.20	3.630	1.045	
	1.14	2.23	3.628	1.045	
	0.48	2.20	3.625	1.043	
	0.35	2.20	3.624	1.043	
Carbon	5.50	2.25	2.973	0.834	
Tetra-Chloride	4.22	2.26	2.969	0.833	0.828
	2.50	2.25	2.964	0.831	
	1.10	2.24	2.963	0.831	
	0.50	2.22	2.960	0.830	
	0.34	2.25	2.960	0.830	
Benezene	5.30	2.28	2.070	0.568	
	4.10	2.30	2.068	0.568	0.561
	2.45	2.30	2.065	0.567	
	1.20	2.28	2.064	0.566	
	0.47	2.28	2.062	0.566	
	0.36	2.30	2.062	0.566	

In order to find the effect of the spacing the liquid below the inner cylinder and the bottom of the outer cylinder on the logarithmic decrement and consequently on the viscosity, a detailed study was made for some liquids for spacings between 5.5 mm. and 0.35 mm. The results are given below.

From the above table, it is easily seen that the change in viscosity is only in the third decimal place and the value of η is practically constant and reached the standard value when the spacing between the bottom of the cylinders is about 2mm.

DISCUSSION

Above is described a method of measuring the viscosity of a liquid by measuring the logarithmic decrement of the rotational motion of a sphere or a cylinder immersed in a liquid. This method can be used for all liquids, viscous as well as mobile and is absolute. It is convenient for measuring the effect of temperature, as the apparatus is of small dimensions, unlike the previous types used for the purpose.

In order to get the sphere and cylinder to execute harmonic angular oscillation and to get a reasonable value for the logarithmic decrement, for very viscous liquid, the oscillating masses must be large. They must be made of steel or brass. For mobile liquids, aluminium can be used. It was shown experimentally that the energy taken up by the layers of liquid below the inner cylinder is negligible. Recently an estimate of this was given by Roscoe (1962). But it can be used only in the case of cylinder in uniform rotation.

REFERENCES

Choudhary, K. R. and Trivedi, G. K., 1961, *J. Sci. Indust. Res.* **20***B*, 470.

Christophorson, D. C., Gemart, A., Hogg, H. A. and South Well, R. M., 1938, *Proc. Roy. Soc.* **163***A*, 351.

Helmholtz, H. V. and Peotrowski, G. V., 1860. *Wien. Bar*, **50**, 107.

Lamb A (1) 1916 Hydrodynamics, 575.

Lamb A (2) -do- 582.

„ (3) -do- 580.

Mennerotte, M., 1911, *J. Phys. Radium* **1**, 753.

Meyer O. E., 1891, *Wied. Amn.* **43**.

Searle G. F., Champton and Davy—Properties of Matter 251.

Stokes G. G., Champion, F. C. and Davy, N. Properties of Matter, 253.

Venkataraman, S., 1933, *Ind. Jour. Phys.* **8**, 26.